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Verifying that the utility function is truly a utility function:
The non-satiation principle means that as you increase consumption of a good, utility increases. This is written mathematically as $MU_X \equiv \partial U/\partial X > 0$. The diminishing marginal utility means that $\partial(MU_X)/\partial X = \partial^2 U/\partial X^2 \equiv U''_{XX} < 0$. ($U''_{XX}$ means you take the derivative with respect to X and then take the derivative of that result with respect to X again.) Repeat these tests for all variables. For example, testing the utility function: $U(X, Y) = X^{1/3} + Y^2$, we find that $U'_X = (1/3)X^{-2/3} > 0 \ \forall\ X, Y > 0$, $U''_{XX} = (-2/9)X^{-5/3} < 0 \ \forall\ X, Y > 0$, $U'_Y = 2Y > 0 \ \forall\ X, Y > 0$, $U''_{YY} = 2 > 0 \ \forall\ X, Y > 0$. Therefore, it passed the first three tests, but failed the last one. It is not a valid utility function. However, $U(X,Y) = X^{1/3}Y^{1/4}$ is a valid utility function because $U'_X = (1/3)X^{-2/3}Y^{1/4} > 0 \ \forall\ X, Y > 0$, $U''_{XX} = (-2/9)X^{-5/3}Y^{1/4} < 0 \ \forall\ X, Y > 0$, $U'_Y = (1/4)X^{1/3}Y^{-3/4} > 0 \ \forall\ X, Y > 0$, $U''_{YY} = (-3/16)X^{1/3}Y^{-7/4} < 0 \ \forall\ X, Y > 0$.

Since a bundle of goods can be thought of as one good, the utility functions must have decreasing returns to scale. (Technically, this test will not always work. Rather we should do is a test which involves matrices which will always work. However, it is complicated.) The formal way to check for returns to scale is to multiply all goods by the same constant and see if utility increases by less than that amount. For example, $U=XYZ$ does not qualify as a utility function because if you multiply all goods by 2, you get $U(2X, 2Y, 2Z) = (2X)(2Y)(2Z) = 8XYZ > 2U(X, Y, Z) = 2XYZ$. Since $8 > 2$, that is increasing returns to scale. If $U = (XYZ)^{1/4}$, then you will get $U(2X, 2Y, 2Z) = 2^{3/4}XYZ$, which is less than $2XYZ$. Therefore, this is an acceptable utility function. **However, you should use a constant like "k" instead of a number.** Therefore, the correct way to do this last test is to do $U(kX, kY, kZ) = k^{3/4}XYZ$, which is less than XYZ. Therefore, it is has decreasing returns to scale and is a valid utility function. Note that the value of $\partial^2 U/\partial X\partial Y$ is irrelevant unless we do the other test involving a matrix. ($\partial^2 U/\partial X\partial Y \equiv U''_{XY}$ is the cross second partial derivative and would determine substitutes and complements, but we are not worrying about it in this course.)

A way to simplify your work which may or may not help:
Since utility functions require that $U'_X > 0$ and $U''_{XX} < 0$, we often get functions that are hard to work with. (U''_{XX} is the second derivative with respect to X and U''_{XY} would be the second derivative, obtained by first taking the derivative with respect to X and then with respect to Y.) For example, we might be given utility functions: 1) $U(X,Y) = (X*Y)^{1/5}$ or 2) $U(X,Y) = \ln(XY)$. If you have one of these functions, you can take $F(U(X,Y))$ and maximize F with respect to X and Y as you did before. That is providing that $F > 0$ and $F'_U > 0$ for all values of U in the feasible set. That means that the transformation must yield a positive number and when utility increases, the function of utility increases.

So for equation 1, you can raise it to the fifth power. That will mean $F(U) = U^5 = (X*Y)^{(1/5)*5} = X*Y$. This is allowed because $F > 0$ since $U > 0$ in the first quadrant ($\forall X, Y > 0$). Also, $F'_U = 5U^4 > 0$ for all $U > 0$. However, this method will not work with the equation 2. In order to get the function to be something nice you would want to have $F(U) = U^{-1/2} = (XY)^{(-2)*(-1/2)} = XY$. But $F'_U = (-1/2)U^{-3/2} < 0 \ \forall\ U > 0$. This means that the F function is a decreasing function, which is easily seen by letting U = 1, 4, and 9. F becomes 1, 1/2, and 1/3 respectively.

Once you have the transformation, you can maximize $F(U(X,Y))$ subject to the budget constraint. This will give you: $\mathfrak{L}(X, Y, \lambda) = F(U(X,Y)) + \lambda(I - P_xX - P_YY)$. Take the derivatives with respect to all three variables. In the case of the former equation, if the price of X is \$10/unit, the price of Y is \$5/unit, and your income is \$160, you would have $\mathfrak{L}(X, Y, \lambda) = XY + \lambda(150 - 10X - 5Y)$. Work it just like any other Lagrangian. So you would get

$\mathcal{L}'_x = Y - 10\,\lambda = 0$,
$\mathcal{L}'_y = X - 5\,\lambda = 0$, and
$\mathcal{L}'_\lambda = 160 - 10X - 5Y = 0$.
The first two equations imply that $\lambda = Y/10$ and $\lambda = X/5$. So $Y = 2X$. Substitute them into the last equation and you get $160 - 10X - 10X = 0$. Thus, $X = 8$, $Y = 16$, and $U = (8*16)^{1/5} = 2*4^{1/5} = 2*1.319508 = 2.639016$.

There is one big problem with doing this transformation. The λ in the transformed equation does not have any meaning. It is telling you how much the function of the utility function increases when your income increases. That is meaningless because the F function is meaningless. Therefore, if you are asked to estimate the effect of the change in the value of the constraint on the level of utility, you cannot do this.

Isoquants and Isocost Lines:
Isoquants and isocost lines work the same as indifference curves and budget constraints. However, normally the objective function and the constraint are reversed. For example, with utility functions, the curved lines (indifference curves) are the objective function and the straight line (budget constraint) is the constraint. This is reversed with isoquants and isocost lines because the objective is a low cost and the constraint is the output. This is not a problem. Just set up the Lagrangian equation as $\mathcal{L}(K, L, \lambda) = P_K K + P_L L + \lambda(Q_0 = Q(X,Y))$ where Q_0 is the desired output.

For example, suppose that capital costs \$4/K and labor costs \$9/L. The production function is given by $Q = (KL)^{1/3} = K^{1/3}L^{1/3}$. The Lagrangian is given by $\mathcal{L} = 4K + 9L + \lambda(Q - K^{1/3}L^{1/3})$. This gives the following first derivatives:
$\mathcal{L}'_K = 4 - \lambda(1/3)K^{-2/3}L^{1/3} = 0$. Therefore, $\lambda = 12\,K^{2/3}L^{-1/3}$. (1)
$\mathcal{L}'_L = 9 - \lambda(1/3)K^{1/3}L^{-2/3} = 0$. Therefore, $\lambda = 27\,K^{-1/3}L^{2/3}$. (2)
$\mathcal{L}'_\lambda = Q - K^{1/3}L^{1/3} = 0$. (3)
Equations (1) and (2) imply that $12\,K^{2/3}L^{-1/3} = 27\,K^{-1/3}L^{2/3}$.
Divide both sides by 3 and multiply both sides by $(KL)^{1/3}$ gives $4K = 9L$, so $K = (9/4)L$. (4)
Substitute (4) into (3) and get $Q = ((9/4)L^2)^{1/3}$. Therefore, $L^{2/3} = (4/9)^{1/3}Q$ and $L = (4/9)^{1/2}Q^{3/2} = (2/3)Q^{3/2}$. From (4), this gives us: $K = (3/2)Q^{3/2}$. Substituting them back into the cost function, we get: $C = 4(3/2)Q^{3/2} + 9(2/3)Q^{3/2} = 6Q^{3/2} + 6Q^{3/2} = 12Q^{3/2}$.
Notice that the costs grow faster than the output, as seen by the exponent on Q's being greater than one. This means the marginal cost curve is upward sloping, as seen by $dC/dQ = 18Q^{1/2}$ which increases as Q increases. The reason for the increasing marginal costs, is that the production function has decreasing returns to scale. You can verify this by multiplying all inputs by some constant, c, and noticing that your new Q is $c^{1/3}(KL)^{1/3}$, which is less than $c(KL)^{1/3}$.

In case you are wondering why I wrote dC/dQ rather than $\partial C/\partial Q$, it is because C was written at that point as a function of only one variable. Thus, it is a full derivative rather than a partial derivative. "Partial derivative" (∂) means you are only taking a derivative with respect to "part" of the equation, i.e., only one of many variables is involved.

When using the minimum cost method of getting the total cost, marginal cost, and average cost, (subject to the constraint that the production function equals the desired output), the λ is the approximation to the MC, but not exactly equal to it.